Exam Mechanics & Relativity 2017–2018
WPPH16006 (part Ib) and NAMR2
January 26, 2018

INSTRUCTIONS

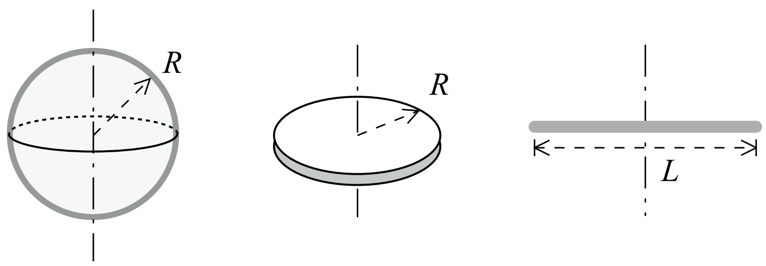
- Use a black or blue pen.
- Write the name of your tutor and/or group on the top right-hand corner of the first sheet handed in, and *deposit your work in the box with your tutor's name.*
- This exam comprises 4 problems. The first three problems require clear arguments and derivations, all written in a well-readable manner, the last one is a multiple choice question.
- The total number of points per problem is

Problem	# of points
1	8
2	9
3	6
4	4

and the grade is computed as $(\text{total \# points})/27 * 9 + 1$.

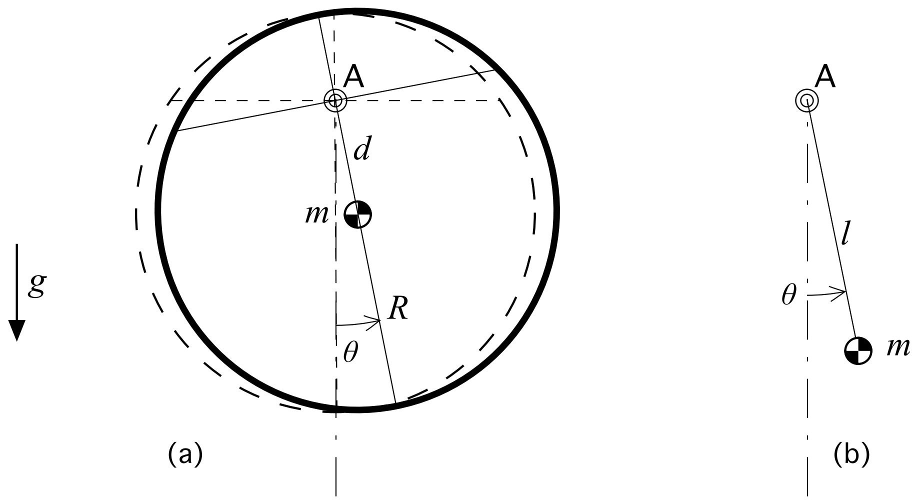
Some useful formulas

The mass of the thin-walled sphere, the thin disk and the rod shown below is M and distributed uniformly. The moments of inertia about the axis (shown by the dashed-dotted line) through their center of mass are:



$$I = \frac{2}{3}MR^2 \qquad I = \frac{1}{2}MR^2 \qquad I = \frac{1}{12}ML^2$$

Problem 1 (8 points)
A thin ring of mass m and radius R is connected to a frictionless hinge by way of a massless cross. The axis of rotation is perpendicular to the plane of the ring. The hinge is at a distance d above the center of the ring, see figure (a).



a. Determine the moment of inertia of the ring about the hinge.

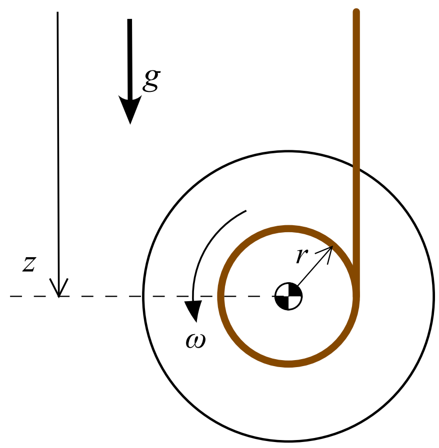
b. Derive the equation of motion for small excursions θ of the ring around the equilibrium position (indicated by the vertical dash-dotted line). Express the period in terms of the quantities indicated in figure (a).

Next, consider the pendulum in figure (b), consisting of a point mass m on a massless string with length l. The question we wish to address is if this simple pendulum can have the same period as the ring in (a).

c. Show that the condition for which both periods are identical is a purely geometrical one (i.e., it only involves quantities with physical dimension of length).

d. Not every ratio of d and R allows for a solution of this relation. What is the smallest value of l/R for which the periods are the same? In that case, what is the value of d/R?

Problem 2 (9 points)
A yo-yo consists of two disks connected by a cylindrical axle of radius r. An inextensible string is looped around the axle from one end and held by hand on the other end. When you let go of the yo-yo, it will start spinning under the action of gravity, will roll downwards until the yo-yo reaches the end of the string and, if you do it well, the yo-yo will wind up towards your hand. The mechanics of how this works is quite interesting. But let's keep it simple and just study what happens when the yo-yo is running down. We will assume that the string remains vertical and neglect all possible sources of friction. The moment of inertia of the entire yo-yo about the center of mass is I, its mass is m.



a. Show that, at any moment, the relation between the rate of change of the position z and the instantaneous angular velocity ω is given by

$$\dot{z} = r\omega\,.$$

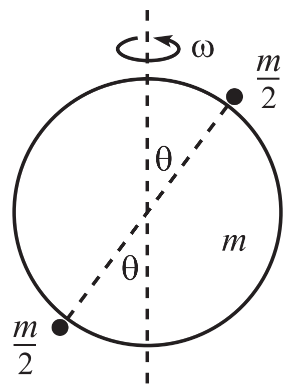
b. Express the total kinetic energy of the yo-yo in terms of the downward velocity $\dot{z}$.

c. Use conservation of energy to show that the acceleration of the yo-yo is given by

$$\ddot{z} = g\frac{m}{m + I/r^2}\,.$$

d. Make a free-body diagram of the yo-yo and determine the tension in the string, S.

Problem 3 (6 points)
A solid sphere of radius R with a uniform mass m rotates about a vertical axis with constant spin ω. Two masses, $m/2$ each, are placed close to the sphere opposite to each other at an angle θ with the vertical axis (see figure). The masses are at rest initially and are then stuck to the (spinning) sphere. As a result, the spin vector will change.



a. Show that the moment of inertia of a solid sphere through its center is equal to $\frac{2}{5}mR^2$. Hint: make use of the moment of inertia of a thin-walled sphere given on the front page of the exam.

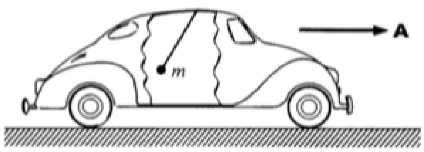
b. Determine the principal axes of the system sphere+masses and demonstrate that the ratio between the largest and the smallest principal moment of inertia is 7/2.

c. Compute the components of the spin vector with respect to these principal axes after the masses have been attached.

Problem 4 (Multiple choice questions: only give answers (A thru D), not reason why) ($2 + 2 = 4$ points)

a. A weight is attached to the ceiling of a car by a string. When the car is at rest, so is the mass and the string hangs vertically downward. When the car accelerates, however, this is no longer the case, even when we neglect relativistic effects.

Two observers, one inside the car and the other by the side of the road, are somehow measuring both the tension in the string and the angle between the string and the vertical. Which of the following statements should they report? More than one answer is possible.



A) The tension in the string is the same for both observers.

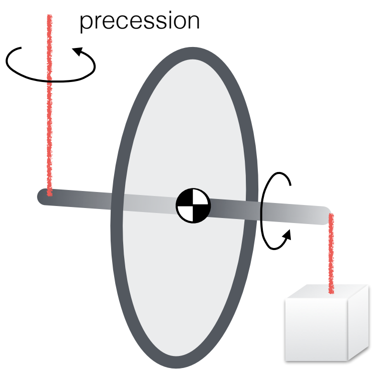
B) The two observers measure different values of the tension in the string.

C) The angle observed by both observers is the same.

D) The angle observed by the two observers is different.

b.

A gyroscopic wheel is supported at one end and while it spins about its axis, it precesses with angular velocity Ω. When you hang a weight on the opposite side of the axle, as indicated on the right, how does this affect precession?



A) The precession frequency remains the same.

B) The precession frequency increases.

C) The precession frequency decreases.

D) The wheel tips over.

Answers Mechanics & Relativity 2017–2018
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Problem 1

a. $I_A = mR^2 + md^2$. 1

b. $T = 2\pi/\omega = 2\pi\sqrt{I_A/mgd}$ 3

c. $\dfrac{R^2 + d^2}{d} = l$ 2

d. The smallest real-valued solution to

$$\left(\frac{d}{R}\right)^2 - \left(\frac{l}{R}\right)\left(\frac{d}{R}\right) + 1 = 0$$

is obtained for $l/R = 2$, namely $d/R = 1$. 2

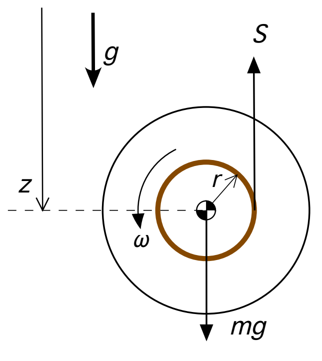
Problem 2

a. At any moment, the instantaneous velocity of the contact point is $= 0$ and just as for a rolling wheel, the velocity of the center is just ω times distance. Q.E.D. 1

b. $T = \frac{1}{2}(m + I/r^2)\dot{z}^2$. 2

c. $\ddot{z} = g\dfrac{m}{m + I/r^2}$ 3

d.



$$S = \frac{mg}{1 + mr^2/I}$$

3

Problem 3

a.
$$I = \int_0^r \frac{2}{3} r^2 dm = \int_0^r \frac{2}{3} r^2 \frac{m}{\frac{4}{3}\pi a^3} 4\pi r^2 dr = \frac{2}{5} ma^2.$$
2

b. The x-axis along the line joining the two masses, and the y axis perpendicular to this are principal axes, with corresponding principal moments

$$I_x = (2/5)mR^2 \text{ and } I_y = (2/5)mR^2 + 2(m/2)R^2 = (7/5)mR^2\,,$$

giving $I_y/I_x = 7/2$ as required. 2

c. $\boldsymbol{\omega}' = (L_x/I_x, L_y/I_y) = (\omega\cos\theta, \frac{2}{7}\omega\sin\theta)$ 2

Problem 4

a. A, C 2

b. B 2